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Fluctuations in the Curie-Weiss Version of the Ising Model with Random Field.

J. M. G. AMARO DE MATOS(*) and J. FERNANDO PEREZ(**)

Instituto de Física, Universidade de São Paulo
C. P. 20516, 01498 São Paulo, SP, Brazil

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Abstract. – For the Curie-Weiss version of the Ising Model with random field it is shown that the fluctuations have: i) a Gaussian distribution with random (*i. e.* sample dependent) mean, if the system is away from criticality or at first-order critical points; ii) a sample-independent non-Gaussian distribution at second- or higher-order critical points.

1. Introduction.

In a beautiful series of papers [1-3], Ellis and Newman discussed the statistics of the large spin-block variables in classical Curie-Weiss models of spin systems and showed their fluctuations to be of nontrivial nature at second-order critical temperatures. It is, therefore, natural to investigate how the behaviour of these variables is affected by the presence of randomness. Of special interest is the so-called Ising model with random magnetic field whose thermodynamics and phase diagram in its Curie-Weiss version have already been computed by Salinas and Wreszinsky [4].

In this paper we revisit the model with the purpose of discussing the probability distribution of its fluctuation variables in the spirit of Ellis and Newman's ideas. In particular, we are interested in questions concerning the self-averaging properties [5]. Our findings are as follows.

Away from criticality, the fluctuations are non-self-averaging. Their probability distribution is a Gaussian with random mean, *i.e.* not determined with probability one in the space of the h_i's. Actually the mean itself is a Gaussian random variable.

At criticality two different kinds of phenomena may occur. If there is a first-order phase transition, fluctuations will remain non-self-averaging, just as above. If not, they will become deterministic and no longer Gaussian. An eventual tricritical point would fall in this last category.

A suggestive physical picture may be drawn from these results, considering the non-self-

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averaging effect due to the presence of the random fields. At a second-order phase transition the correlations between spins, typical of criticality, are strong enough to suppress the effect of field's fluctuations.

We describe here the main ideas leading to our results. Full mathematical details and applications to other Curie-Weiss random spin models, like Van Hemmen's spin-glass model [6], will be presented elsewhere [7].

The paper is organized as follows. In the next section we present the model and briefly review its thermodynamics within a slightly different method than that used by Salinas and Wreszinski. In the third section we present the analysis of the fluctuations and obtain our results.

2. The model and its thermodynamics.

The model is described by the following Hamiltonian:

$$H_n = -\frac{1}{2n}\left(\sum_{i=1}^{n}\sigma_i\right)^2 - \sum_{i=1}^{n} h_i\sigma_i, \tag{2.1}$$

where $\sigma_i = \pm 1$ are spin variables and the fields h_i are independent, identically distributed random variables with distribution $\mathrm{d}\nu(h_i)$, that we denote by $h_i \sim \mathrm{d}\nu(h_i)$.

Denoting by $\{\sigma\}$ all the possible configurations of spins, its partition function may be written

$$Z_n = \sum_{\{\sigma\}} \exp[-\beta H_n] = \left(\frac{n}{2\pi}\right)^{1/2} 2^n \int \mathrm{d}x \exp\left[-n\frac{x^2}{2}\right] \prod_{i=1}^{n} \exp[\ln\cosh\sqrt{\beta}(x+\sqrt{\beta}h_i)] =$$

$$= 2^n\left(\frac{n}{2\pi}\right)^{1/2}\int \mathrm{d}x \exp[-n\,G_n(x)]. \tag{2.2}$$

Here we have used the identity

$$\exp[a^2/2] = \frac{1}{\sqrt{2\pi}}\int \exp[-x^2/2 + ax]\,\mathrm{d}x,$$

in the first equality with

$$a = \sqrt{\beta/n}\sum_{i=1}^{n}\sigma_i,$$

and

$$G_n(x) = \frac{x^2}{2} - \frac{1}{n}\sum_{i=1}^{n}\ln\cosh\sqrt{\beta}(x+\sqrt{\beta}h_i) \underset{n\to\infty}{\longrightarrow} G(x) = \frac{x^2}{2} - \overline{\ln\cosh\sqrt{\beta}(x+\sqrt{\beta}h)}$$

by the law of large numbers, where

$$\overline{\ln\cosh\sqrt{\beta}(x+\sqrt{\beta}h)} = \int \mathrm{d}\nu(h)\ln\cosh\sqrt{\beta}(x+\sqrt{\beta}h).$$

Following Laplace's asymptotic method, the free energy $f = \lim_{n\to\infty} -\frac{1}{n\beta}\ln Z_n$ will be given by

$$f = \inf_{x\in R} G(x) = G(x^*).$$

An important point should be noticed. If one computes Z_n with a fixed configuration of fields $\{h_i\}$, one obtains a free energy independent of the choice of $\{h_i\}$, since the thermodynamic limit itself provides an average on $d\nu(h)$. This is the self-averaging property of the free energy.
Denoting by $G^{(k)}(x)$ and $G_n^{(k)}(x)$ the k-th derivatives of $G(x)$ and $G_n(x)$, respectively, one may write for the first derivatives of $G(x)$,

$$\begin{cases} G^{(1)}(x) = x - \sqrt{\beta}\int d\nu(h)\,\mathrm{tgh}\sqrt{\beta}(x+\sqrt{\beta}h)\,, \\ G^{(2)}(x) = 1 - \beta\int d\nu(h)\,\mathrm{sech}^2\sqrt{\beta}(x+\sqrt{\beta}h)\,. \end{cases} \tag{2.3}$$

The derivatives of G_n will be given by similar expressions, just replacing $\int d\nu(h)$ by $\frac{1}{n}\sum_i$ and h by h_i.
Clearly $G^{(1)}(x^*) = 0$ and for any even distribution $d\nu(h)$ all the odd derivatives of $G(x)$ vanish at $x = 0$. So, in this case $x^* = 0$ will be a (at least local) minimum as long as $G^{(2)}(0) > 0$. For $\beta = 0$ one can see that it is the only global minimum. As β increases, if there is no first-order phase transition, the condition $G^{(2)}(0) = 0$ defines a second-order phase transition at $\beta = \beta_c$ given by (2.3):

$$\beta_c = \left[\int d\nu(h)\,\mathrm{sech}^2\beta_c h\right]^{-1}.$$

We will consider two types of distribution:

Type I: even, absolutely continuous with density $p(h)$ decreasing in $[0, \infty]$.
Type II: density of the form $p(h) = C_0\delta(h) + \sum_i C_i[\delta(h-h_i)+\delta(h+h_i)]$ with $C_i \geq 0$ for all i.

It may be shown [4] that for $d\nu$ of type I there will be no first-order phase transition and neither for type II if $C_0 - (1/3)\sum_i C_i \geq 0$. If, on the other hand, one takes

$$p(h) = \frac{1}{2}[\delta(h-H) + \delta(h+H)]\,,$$

one finds a phase diagram as depicted in fig. 1 [4], where (H_t, T_t) is a tricritical point.

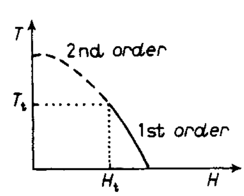

Fig. 1.

3. Fluctuations.

The study of fluctuations should be regarded as a finite-size statistical correction to the evaluation of the equilibrium magnetization. We are then interested in the asymptotic distribution of the random variable y_n defined as

$$y_n = \frac{\sum_i \frac{\sigma_i}{n} - m^*}{n^{-\gamma}}$$

with $\gamma > 0$ and to be chosen in such a way that $y = \lim_{n\to\infty} y_n$ has a nontrivial probability distribution.

Moreover, if $y_n \sim P_n(x)\,\mathrm{d}x$, then P_{2n} is given [8] by

$$P_{2n} = \int \mathrm{d}x'\, P_n(x')\, P_n(a^{-1}x - x')\, g_n(x, x', a)\,,$$

where $g_n(x, x', a)$ expresses the interactions between the σ_i and a is some constant related to γ. Then $P(x) = \lim_{n\to\infty} P_n(x)$ will be stable [9] in the sense that it satisfies the fixed-point equation

$$P(x) = \lim_{n\to\infty} \int \mathrm{d}x'\, P(x')\, P_n(a^{-1}x - x')\, g_n(x, x', a)\,.$$

One should take very carefully into account that we are considering y_n as a block-spin variable of the size of the system with n (finite) fixed. This crucial point is in the heart of the whole approach, for had one considered a block-spin variable of *fixed* size (say N) in the thermodynamic limit ($n \to \infty$), its statistic would have been trivial (Gaussian distribution), since the spin variables are asymptotically independent. All nontriviality of our results to be exposed hereafter comes from this very fact. Here, the role of the function $G(x)$ becomes clearer. Its origin is an integral representation for Z_n, where the spins are uncoupled already for finite n. The cost of such an operation is the introduction of the real, continuous variable x in (2.2). It appears as an extra field in a modified Hamiltonian of independent spins, with a Gaussian weight in the measure under integration. In fact, as one easily sees [2], the function $\exp[-nG(x)]$ admits a decomposition as a convolution of a Gaussian distribution with that of y_n:

$$\frac{W}{n^{1/2-\gamma}} + y_n \sim \exp\left[-n\, G_n\!\left(\frac{s}{n^{\gamma}} + m^*\right)\right], \tag{3.1}$$

where W is a random variable independent of y_n and $W \sim N(0, 1)$, with $N(0, 1)$ being a Gaussian distribution with mean zero and variance 1. This fact allows one to obtain the asymptotic distribution in (3.1) by expanding $G_n(s/n^{\gamma} + m^*)$ around its minimun and then taking the thermodynamic limit. If G_n has several, say p, minima given by $\{x_n^{(i)}\}\ i = 1, 2 \ldots p$, the distribution obtained will correspond to conditioning the magnetization to a neighbourhood of the point around which one is expanding G.

For the class of measures we considered, we can state our results in the following form:

$$y \sim \begin{cases} \exp\left[-G^{(6)}(0)\dfrac{s^6}{6!}\right]\mathrm{d}s, & \text{at the tricritical point, if it exists,} \\[2ex] \exp\left[-G^{(4)}(0)\dfrac{s^4}{4!}\right]\mathrm{d}s, & \text{at the critical point of a 2nd-order phase transition,} \\[2ex] N\left(\alpha, \dfrac{1}{G^{(2)}(0)} - 1\right), & \text{at any other temperature where } \alpha \sim N(0, \sigma^2)\,, \\ & \text{with } \dfrac{\sigma^2}{\beta} = \int \mathrm{d}\nu(h)\,\mathrm{tgh}^2\sqrt{\beta}(x^* + \sqrt{\beta}\,h) - \left[\int \mathrm{d}\nu(h)\,\mathrm{tgh}\sqrt{\beta}(x^* + \sqrt{\beta}\,h)\right]^2. \end{cases}$$

This can be seen from the expansion of $G_n(x)$ in the form

$$n\,G_n\left(\frac{s}{n^\gamma}\,x + x^*\right) = \sum_{k=0}^{\infty} G_n^{(k)}(x_n^*)\,[s - n^\gamma(x_n^* - x^*)]^k\,\frac{n^{1-k\gamma}}{k!}, \qquad \text{as } s \to n^\gamma(x_n^* - x^*),$$

where $G_n^{(0)}(x) = G_n(x)$. At the tricritical point all the derivatives before the sixth vanish, so one must take $\gamma = 1/6$ in order to save the expansion as $n \to \infty$. This gives our first result. The second comes from a similar argument, since the criticality of a 2nd-order phase transition is characterized by $G^{(2)}(0) = 0$. In that case one must take $\gamma = 1/4$. Finally, if $G^{(2)}(0) > 0$ one must take $\gamma = 1/2$ and the random variable $\alpha = \sqrt{n}\,(n_n^* - x^*)$ has the distribution given above by the central-limit theorem.

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