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1992 Europhys. Lett. 18 661

(http://iopscience.iop.org/0295-5075/18/7/015)

EUROPHYSICS LETTERS 1 April 1992

Europhys. Lett., 18 (7), pp. 661-664 (1992)
Erratum

Fluctuations in the Curie-Weiss Version of the Ising Model with Random Field (*Erratum*).

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(received 20 December 1991; accepted 9 March 1992)

PACS. 02.50 – Probability theory, stochastic processes, and statistics.
PACS. 05.50 – Lattice theory and statistics; Ising problems.
PACS. 05.70 – Thermodynamics.

In the letter published in 1988 [1] the correct statements concerning the fluctuations of the magnetization in the Curie-Weiss version of the random field Ising model should read as follows:

1) a Gaussian distribution with random (*i.e.* sample-dependent mean) if the system is away from criticality or at a first-order critical point;

2) a sample dependent non-Gaussian distribution at a second- and higher-order critical point.

Moreover, in the latter case the thermal fluctuations turn out to be always dominated by the fluctuations originated in the randomness of the field, as already assumed in the literature [2].

Using the same notation as in [1] the correct distribution of the variable $y \equiv \lim_{n \to \infty} y_n$ with $y_n = n^{\gamma}(\sum \sigma_i/n - m^*)$ and $\beta \leq \beta_{2c}$, $m^* = 0$ is

1) $s^4 \exp[-[G^{(6)}(0)/5!\sigma]^2 (s^{10}/2)]\,ds$ with $\gamma = 1/10$ at the tricritical point, if it exists;

2) $s^2 \exp[-[G^{(4)}(0)/3!\sigma]^2 (s^6/2)]\,ds$ with $\gamma = 1/6$ at the second-order criticality;

3) $N(a, 1/G^{(2)}(0) - 1)$, with $\gamma = 1/2$ away from criticality or at a first-order critical point with $a \sim N(0, \sigma^2)$, where

$$\sigma^2 = \beta \left\{ \int d\nu(h)\, \mathrm{tgh}^2(\beta h) - \left[\int d\nu(h)\, \mathrm{tgh}(\beta h) \right]^2 \right\}.$$

A rigorous proof of these results may be found in [3,4]. The heuristics of the derivation

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goes as follows. We start with

$$\frac{1}{\sqrt{\beta}}\frac{W}{n^{1/2-\rho}}+z_n \sim \exp\left[-nG_n\left(\frac{\sqrt{\beta}s}{n^{\rho}}+x_n^*\right)\right], \tag{1}$$

where $z_n \equiv n^{\rho}[\sum \sigma_i/n - x_n^*/\sqrt{\beta}]$, and x_n^* is the minimum of $G_n(x)$. We then expand G_n around $s=0$ to obtain

$$nG_n\left(\frac{\sqrt{\beta}s}{n^{\rho}}+x_n^*\right)=\sum_{l=0}^{\infty} G_n^{(l)}(x_n^*)(\sqrt{\beta}s)^l\frac{n^{1-l\rho}}{l!}.$$

It is important now to control the rate of convergence of $G_n^{(l)}(x_n^*)$ to $G^{(l)}(x^*)$. To that extent we shall first control the rate of convergence of x_n^* to x^*. This is done by considering the equations

$$x_n^* = \frac{\sqrt{\beta}}{n}\sum_{i=1}^{n}\operatorname{tgh}[\sqrt{\beta}(x_n^*+\sqrt{\beta}h_i)], \tag{2}$$

$$x^* = \sqrt{\beta}\int d\nu(h)\operatorname{tgh}[\sqrt{\beta}(x^*+\sqrt{\beta}h)], \tag{3}$$

the first of which is a sum of *highly dependent* random variables; hence the central-limit theorem is not directly applicable. It is however possible to compute the asymptotic behaviour:

$$n^{\delta}(x_n^*-x^*)=v, \tag{4}$$

where the random variable v is distributed according to the density

$$s^{2k-2}\exp\left[-\left[\frac{G^{(2k)}(0)}{(2k-1)!\,\sigma}\right]^2\frac{s^{2k-1}}{2}\right]ds \tag{5}$$

with $k=1$ away from criticality or at first-order critical points, $k=2$ at second-order criticality, $k=3$ at tricriticality and so on. Also, the power δ is related to k by $\delta = 1/2(2k-1)$.

Notice that v is a random variable that depends on the configuration $\boldsymbol{h}=\{h_i, i\in\boldsymbol{Z}\}$ of the external fields, and so the fluctuations are absent in the deterministic case ($h_i=0, i\in\boldsymbol{Z}$) being of nonthermal origin.

We now use the fact that $G^{(l)}(0)$ are sums of independent identically distributed random variables to get from the central-limit theorem

$$G_n^{(l)}(0)\simeq G^{(l)}(0)+\frac{u_l}{\sqrt{n}},$$

where u_l are Gaussian random variables. Using this result together with eqs. (2) and (3), we compute

$$nG_n\left(\frac{\sqrt{\beta}s}{b^{\rho}}+x_n^*\right)=n\left\{G(0)+\frac{\beta s^2}{2n^{2\rho}}\left[G^{(2)}(0)+\frac{G^{(4)}(0)}{2}(x_n^*)^2+\frac{u_2}{\sqrt{n}}+\ldots\right]+\ldots\right\}.$$

Making use of this expansion, we control the right-hand side of (1) in the limit $n\to\infty$ with the choice $\rho = k\delta$. As a consequence, $\rho \geq \delta$. Therefore, for large n,

$$\frac{\sum \sigma_i}{n} \simeq \frac{x_n^*}{\sqrt{\beta}} + \frac{z_n}{n^\rho} = \frac{v}{n^\delta} + \frac{z_n}{n^\rho}$$

and so, if $\delta = 1/2$, $\rho = 1/2$ (*i.e.* away from criticality or at a first-order criticality) both terms contribute in the same order. However, at a second-order critical point, $\delta = 1/6$ and $\rho = 1/3$. This implies that

$$\frac{\sum \sigma_i}{n} \simeq \frac{v}{n^\rho},$$

and the result follows.

Let us now discuss the rate of convergence of x_n^* to x^* in the particularly simple case $h_i = \pm H$ with equal probability. In this case

$$g_n(x) = \frac{x^2}{2} - f_n^+(\boldsymbol{h})\,\varphi_+(x) - f_n^-(\boldsymbol{h})\,\varphi_-(x),$$

where $f_n^\pm(\boldsymbol{h}) = (1/n)\sum(H \pm h_i)/2H$ = fraction of sites where h_i equals $\pm H$, respectively, and $\varphi_\pm(x) = \ln\cosh(\sqrt{\beta}x \pm \beta H)$.

In this case eq. (2) which gives the condition of a minimum at x_n^* reads

$$x_n^* = f_n^+(\boldsymbol{h})\,\varphi'_+(x_n^*) + f_n^-(\boldsymbol{h})\,\varphi'_-(x_n^*). \tag{6}$$

Let now $x^* = 0$ be the minimum of $G(x) = \lim_{n\to\infty} G_n = x^2/2 - (1/2)[\varphi_+(x) + \varphi_-(x)]$. Expanding the right-hand side of (6) in powers of x_n^* around zero, we have

$$x_n^* \simeq [f_n^+\varphi'_+(0) + f_n^-\varphi'_-(0)] + [f_n^+\varphi''_+(0) + f_n^-\varphi''_-(0)]\,x_n^* +$$
$$+ [f_n^+\varphi_+^{(3)}(0) + f_n^-\varphi_-^{(3)}(0)]\frac{(x_n^*)^2}{2} + [f_n^+\varphi_+^{(4)}(0) + f_n^-\varphi_-^{(4)}(0)]\frac{(x_n^*)^3}{3!} + \dots.$$

Defining $\psi(x) \equiv \varphi_+(x)$, we notice that by construction,

$$\varphi_+^{(l)}(0) + (-1)^{l+1}\varphi_-^{(l)}(0) = 0$$

and so

$$x_n^*[1 - (f_n^+ + f_n^-)\,\psi''(0)] \simeq (f_n^+ - f_n^-) + (f_n^+ - f_n^-)\frac{\psi^{(3)}(0)}{2}(x_n^*)^2 + (f_n^+ + f_n^-)\frac{\psi^{(4)}(0)}{3!}(x_n^*)^3 \dots.$$

Since each f_n is a sum of identical independently distributed random variables and $f_n^+ - f_n^- \to 0$ by construction, the first term of the right-hand side dominates all the others, since it goes to zero as $u/\sqrt{n}$, where u is a Gaussian random variable. If the left-hand side is different from zero, *i.e.* if $\psi''(0) < 1$, it then follows that x_n^* goes to zero as $1/\sqrt{n}$, which implies $\delta = 1/2$, and v is Gaussian. However, at the second-order criticality, when $\psi''(0) = 1$ and $\psi^{(3)}(0) > 0$, the left-hand side is zero and we can solve the equation for the leading term in $(x_n^*)^3$ to get

$$(x_n^*)^3 \to \frac{3!\,u}{\sqrt{n}\,\psi^{(4)}(0)} \quad \text{as } n \to \infty .$$

This implies $\delta = 1/6$ and $v = (3!\,u/\psi^{(4)}(0))^{1/3}$ as stated above in eq. (5).

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